9

PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

Art-1. Order and Degree and Formation of Partial Differential Equations

Definition 1. When a differential equation contains one or more partial derivatives of an unknown function of two or more variables (independent) ; then it is called a **Partial differential Equation.**

Note. (*i*) We consider, generally x and y as independent variables (in case of two variables) and z as dependent variable *i.e.,* $z = f(x, y)$

(*ii*) We denote partial derivatives of first and higher orders as

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x\, \partial y}$$

for which symbols p, q, r, t, s respectively will be used in this topic

i.e., $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x\, \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$

Definition (2) :– The **order** of a partial differential equation is defined as the order of the highest partial derivative occuring in it and **the degree** is defined as the exponent of the highest order partial derivative.

Definition (3) :– A partial differential equation is said to be **Linear** if the dependent variable and its partial derivatives occur only in first degree and are not multiplied together in the differential equation. Otherwise the equation is called **Non-linear** (Not linear Equation) differential equation.

EXAMPLES

1. $\frac{\partial z}{\partial x} - 5\frac{\partial z}{\partial y} = 2z + \sin(x - 2y)$; order = 1, degree = 1 and it is Linear.

2. $xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = 3xy$; order = 1, degree = 1 and it is non-linear.

3. $(x^2 - z^2)\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = -3xy$; order = 1, degree = 1 and it is non-linear.

4. $5\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\, \partial y} + 3\frac{\partial^2 z}{\partial y^2} = 0$; order = 2, degree = 1 and it is linear

5. $z\frac{\partial z}{\partial x} + 5\frac{\partial z}{\partial y} = 2y$; order = 1, degree = 1 and it is non-linear.

6. $x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$; Order = 2, degree = 1 and it is linear.

7. $z\left(\frac{\partial z}{\partial x}\right)^2+x^2=3$; Order = 1, degree = 2 and it is non-linear

Art-2. To Form a partial Differential Equation

In general, there are two ways for derivation of partial differential equations.

(*i*) **By eliminating arbitrary constants from the given relation betwe variables.**

(*ii*) **By eliminating arbitrary functions from the given relation betwe variables.**

Art-3. To Form a partial differential Equation by elimination of arbitrary constan

Consider z be function of two independent variables x and y, defined as

$$f(x, y, z, a, b)=0 \quad ...(i)$$

where a and b are arbitrary constants.

Differentiating (*i*) partially w.r.t. x,

we get $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=0$ or $\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0$ ...(i

Also Differentiating (*i*) partially w.r.t. y,

we get $\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=0$ or $\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0$ ...(ii

Now eliminate a, b from (*i*), (*ii*), (*iii*)

we get an equation, say of the form

$$g(x, y, z, p, q)=0$$

which is required partial differential equation of order one.

Note. (*i*) If there are more arbitrary constants than the number of independent variables then the elimination of constants usually shall give rise to a partial differential equation o higher order than one.

(*ii*) If there are less arbitrary constants than the number of independent variables then the elimination of constants usually shall give rise to more then one differential equations of first order.

For example If $z=x+by$

Then differential equations are $q=\frac{z-x}{y}$ and $p=1$.

(*iii*) If the number of arbitrary constants is equal to the number of independent variables, then the elimination of constants usually shall give rise to one differential equation of first order.

ILLUSTRATIVE EXAMPLES

Example 1. Form partial differential equations by eliminating arbitrary constants from the following relations.

(i) $z = (2x + a)(2y + b)$

(ii) $z = ax + by + a^3 + b^3$

(iii) $z = ax + by + ab$

(iv) $z = \frac{1}{3}ax^3 + \frac{1}{3}by^3$

(v) $z = ax + (2 - a)y + b$

(vi) $z = ax + a^2y^2 + b^2$

(vii) $z = \frac{1}{a}(a^2x + y - b)$

(viii) $z = axe^y + \frac{1}{2}a^2e^{2y} + b$

Sol. (i) We are given $z = (2x + a)(2y + b)$ ...(i)

Differentiate partially w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x} = (2y + b)(2 + 0)$ ...(ii)

and $\frac{\partial z}{dy} = (2x + a)(2 + 0)$ ...(iii)

Multiply (ii) by (iii), we get

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = (2y + b)(2)(2x + a)(2)$$

(Using (i))

$\Rightarrow \quad pq = 4z$

which is required partial differentiation equation.

(ii) We are given $z = ax + by + a^3 + b^3$ ...(i)

Differentiate partially w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$

Put these values of a and b in (i)

we get $z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^3 + \left(\frac{\partial z}{\partial y}\right)^3$ or $z = px + qy + p^3q^3$

which is required partial differential equation.

(iii) We are given $z = ax + by + ab$ ...(i)

Differentiate partially w.r.t x and w.r.t y

we get $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$

Put these values of a and b in (i)

we get $z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

or $z = xp + yq + pq$

which is required partial differential equation.

(*iv*) We are given $z = \frac{1}{3}a x^3 + \frac{1}{3} b y^3$

Differentiate partially w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x} = a x^2$ and $\frac{\partial z}{\partial y} = b y^2$

$\Rightarrow \quad p = a x^2$ and $q = b y^2 \quad \Rightarrow a = \frac{p}{x^2}$ and $b = \frac{q}{y^2}$

Put these values of a and b in (*i*)

we get $z = \frac{1}{3}\left(\frac{p}{x^2}\right)x^3 + \frac{1}{3}\left(\frac{q}{y^2}\right)y^3$

$\Rightarrow \quad \mathbf{3z = px + qy}$ **which is required partial differential equation**

(*v*) We are given $z = a x + (2-a) y + b$

Differentiate partially w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = 2 - a$

Eliminating a, we get

$\frac{\partial z}{\partial y} = 2 - \frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2$ or $\mathbf{p + q = 2}$

which is required partial differential equation.

(*vi*) We are given $z = a x + (a^2 y^2 + b^2)$

Differentiate partially w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = 2 a^2 y \quad \Rightarrow p = a$ and $q = 2 a^2 y$

Eliminating a, we get

$q = 2 p^2 y$ or $\mathbf{q = 2 y p^2}$

which is required partial differential equation.

(*vii*) We are given $z = \frac{1}{a}(a^2 x + y - b)$

$z = a x + \frac{1}{a} y - \frac{b}{a}$

Differentiate partially w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = \frac{1}{a}$

$\Rightarrow \quad p = a$ and $q = \dfrac{1}{a}$

Eliminating a, we get $q = \dfrac{1}{p}$ or $\boxed{pq = 1}$

which is required partial differential equation

(*viii*) We are given $z = a\,x\,e^y + \dfrac{1}{2}\,a^2\,e^{2y} + b$

Differentiate partially w.r.t. x and w.r.t. y

we get $\dfrac{\partial z}{\partial x} = a\,e^y$ and $\dfrac{\partial z}{\partial y} = \dfrac{1}{2}a^2\,(2\,e^{2y}) + a\,x\,e^y$

$\Rightarrow \quad p = a\,e^y$ and $q = a^2 e^{2y} + x\,(a\,e^y)$.

$\Rightarrow \quad p = a\,e^y$ ...(*i*)

and $q = (a\,e^y)^2 + x\,(a\,e^y)$ ...(*ii*)

Put $p = a\,e^y$ from (*i*) in (*ii*)

we get $q = p^2 + x\,p$ or $\boldsymbol{q = p\,x + p^2}$ which is required partial diff. equation

Example 2. Find partial differential equation's by eliminating constants from the following relations

(*i*) $z = a\,(x + y) + b$

(*ii*) $z = a\;e^{-b^2 y}\cos b\,x$

(*iii*) $z = a\,e^{-b^2 t}\sin b\,x$

(*iv*) $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$.

(*v*) $z = a\,e^{b\,x}\cos b\,y$

(*vi*) $z = (x^2 + 2\,a)\;(y^2 + 2\,b)$

(*vii*) $z = (x^3 + a)\,(y^3 + b)$

(*viii*) $a\,x^2 + b\,y^2 + z^2 = 1$

(*ix*) $(x - a)^2 + (y - b)^2 + z^2 = r^2$

(*x*) $(z + \alpha^2)^3 = (x + \alpha\,y + \beta)^2$

Sol. (*i*) We are given $z = a\,(x + y) + b$

Differentiate w.r.t x and w.r.t. y

we get $\dfrac{\partial z}{\partial x} = a$ and $\dfrac{\partial z}{\partial y} = a \Rightarrow \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y}$ *i.e.*, $p = q$

which is required differential equation

(*ii*) We are given $z = a\;e^{-b^2 y}\cos b\,x$ ...(*i*)

Differentiate –partially w.r.t. x and w.r.t. y

we get $\dfrac{\partial z}{\partial x} = a\;e^{-b^2 y}\;(-\,b\sin b\,x)$ ...(*ii*)

and $\dfrac{\partial z}{\partial y} = a\,(-\,b^2)\;e^{-b^2 y}\cos b\,x$ ...(*iii*)

Again differential (*ii*) partially w.r.t. x

we get $\dfrac{\partial^2 z}{\partial x^2} = a\, e^{-b^2 y} (-b)(b \cos b x)$ ...(

From (*iii*) and (*iv*), we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$$ (∵ R.H. Sides are sam

or $r = q$ which is required partial differential eq.

(*iii*) We are given

$z = a\, e^{-b^2 t} \sin b x$ ...(

Differential partially w.r.t. t and w.r.t. x

we get $\dfrac{\partial z}{\partial t} = (a)(-b^2)\, e^{-b^2 t} \sin b x$ ...(i

and $\dfrac{\partial^2 z}{\partial x} = a\, e^{-b^2 t} (b \cos b x)$ ...(iii

Again differentiate (*iii*) w.r.t x

we get $\dfrac{\partial^2 z}{\partial x^2} = a\, e^{-b^2 t} . b(-b \sin b x) = (a)(-b^2)\, e^{-b^2 t} \sin b x$ ...(iv)

From (*ii*) and (*iv*), we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}$$ (∵ R.H. Sides are same)

which is required partial differential equation

(*iv*) We are given $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ ...(I)

Differentiate partially w.r.t. x and w.r.t y

we get $\dfrac{\partial z}{\partial x} = \dfrac{2x}{a^2}$ and $\dfrac{\partial z}{\partial y} = \dfrac{2y}{b^2}$ ⇒ $\dfrac{p}{2x} = \dfrac{1}{a^2}$ and $\dfrac{q}{2y} = \dfrac{1}{b^2}$

Put values of $\dfrac{1}{a^2}$ and $\dfrac{1}{b^2}$ in (*i*)

We get $z = \dfrac{px^2}{2x} + \dfrac{qy^2}{2y}$

⇒ $2z = px + qy$

which is required partial differential equation.

(v) We are given $z = a\,e^{bx}\cos by$ ...(i)

Differential partially w.r.t. x and w.r.t y

We get $\frac{\partial z}{\partial x} = a\,b\,e^{bx}\cos by$ ...(ii)

and $\frac{\partial z}{\partial y} = a\,e^{bx}(-b\sin by)$ ...(iii)

$\therefore \quad \frac{(iii)}{(ii)} \Rightarrow \frac{\frac{\partial z}{\partial y}}{\frac{\partial z}{\partial x}} = -\tan by$ ...(iv)

From (i) and (ii) $\frac{\partial z}{\partial x} = bz \Rightarrow p = bz$ or $b = \frac{p}{z}$

Put value of b in (iv)

we get $\frac{\frac{\partial z}{\partial y}}{\frac{\partial z}{\partial x}} = -\tan\frac{py}{z}$ or $\frac{q}{p} = -\tan\frac{py}{z}$ or $q + p\tan\frac{py}{z} = 0$

which is required differential equation

OR In above question, we have

$$\frac{\partial z}{\partial x} = bz$$

Again Differentiate partially w.r.t. x

$$\frac{\partial^2 z}{\partial x^2} = b\,\frac{\partial z}{\partial x}$$

From these two relations, we get $z\,\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial z}{\partial x}\right)^2$

which is also partial differential equation of (i)

OR In above question, we have

From (iii) $\frac{\partial z}{\partial y} = a\,e^{bx}(-b\sin by)$

Again differentiate partially w.r.t. y

$$\frac{\partial^2 z}{\partial y^2} = a\,e^{bx}(-b)\,b\cos by = -b^2(a\,e^{bx}\cos by) = -b^2 z \text{ (by (i))}$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = -b^2 z$$

Art-4. To Form a partial differential equation by eliminating arbitrary functions

Let u and v be two independent functions of three variables x, y, z ; which are given by the relation $f(u, v) = 0$ ...(i)

Differentiating (i) w.r.t. x (taking y as constant)

and w.r.t. y (taking x as constant)

we get $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)=0$$ ...(ii)

$\Rightarrow$ $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$ ...(iii)

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

from (*ii*) $\dfrac{\partial f}{\partial u}\left(\dfrac{\partial u}{\partial x}+p\dfrac{\partial u}{\partial z}\right)=-\dfrac{\partial f}{\partial v}\left(\dfrac{\partial v}{\partial x}+p\dfrac{\partial v}{\partial z}\right)$ ...(*iv*)

and $\dfrac{\partial f}{\partial u}\left(\dfrac{\partial u}{\partial y}+q\dfrac{\partial u}{\partial z}\right)=-\dfrac{\partial f}{\partial v}\left(\dfrac{\partial v}{\partial y}+q\dfrac{\partial v}{\partial z}\right)$ ...(*v*)

On dividing (*iv*) by (*v*) we get

$$\frac{\dfrac{\partial u}{\partial x}+p\dfrac{\partial u}{\partial z}}{\dfrac{\partial u}{\partial y}+q\dfrac{\partial u}{\partial z}}=\frac{\dfrac{\partial v}{\partial x}+p\dfrac{\partial v}{\partial z}}{\dfrac{\partial v}{\partial y}+q\dfrac{\partial v}{\partial z}}$$

Cross multiply, we get

$$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial v}{\partial z}\frac{\partial u}{\partial x}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

$\Rightarrow \quad \mathrm{P}\,q+\mathrm{Q}\,q=\mathrm{R}$ (say) ...(*vi*)

where $\mathrm{P}=\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z}-\dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial z}=\begin{vmatrix}\dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z}\\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z}\end{vmatrix}=\dfrac{\partial(u,v)}{\partial(y,z)}$

and $\mathrm{Q}=\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x}-\dfrac{\partial v}{\partial z}\dfrac{\partial u}{\partial x}=\begin{vmatrix}\dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial x}\end{vmatrix}=\dfrac{\partial(u,v)}{\partial(z,x)}$

and $\mathrm{R}=\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}-\dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial y}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}=\dfrac{\partial(u,v)}{\partial(x,y)}$

$\therefore$ equation (*vi*) is the required differential equation.

ILLUSTRATIVE EXAMPLES

Example 1. Form partial differential equations by eliminating arbitrary functions from the following relations

(*i*) $z=f(x+\lambda y)$ (*ii*) $z=f\left(\dfrac{y}{x}\right)$

(*iii*) $z=f(x^2-y^2)$ (*iv*) $z=f(x^2+2y^2)$

(v) $z = x + y + f(xy)$ (vi) $z = xy + f(x^2 + y^2)$

(vii) $z = f(x) + e^y g(x)$ (viii) $z = f\left(\frac{x}{y}\right) + g(xy)$

(ix) $z = e^{ky} f(x - y)$ (x) $z = f(x + ay) + g(x - ay)$

(xi) $z = f(x + iy) + g(x - iy)$

(xii) $z = f(x^2 - y) + g(x^2 + y)$

(xiii) $z = xf(x + y) + g(x + y)$ (xiv) $z = f(x) + xg(y)$

(xv) $z = yf(x) + xg(y)$

Sol. (*i*) We are given $z = f(x + \lambda y)$ ...(*i*)

Differentiate (*i*) partially w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x} = (f'(x + \lambda y))\,(1 + 0) = f'(x + \lambda y)$ ...(*ii*)

and $\frac{\partial z}{\partial y} = (f'(x + \lambda y))\,(0 + \lambda) = \lambda f'(x + \lambda y)$ ...(*iii*)

Dividing (*iii*) by (*ii*), we get $\frac{\frac{\partial z}{\partial y}}{\frac{\partial z}{\partial x}} = \lambda \Rightarrow \frac{q}{p} = \lambda$

$\Rightarrow \quad q = \lambda p$

which is required partial differential equation.

(*ii*) We are given $z = f\left(\frac{y}{x}\right)$ ...(*i*)

Differentiate partially w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{\partial}{\partial x}\left(\frac{y}{x}\right)\right) = \left(f'\left(\frac{y}{x}\right)\right)\left(-\frac{y}{x^2}\right)$ ...(*ii*)

and $\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{\partial}{\partial y}\left(\frac{y}{x}\right)\right) = \left(f'\left(\frac{y}{x}\right)\right)\left(\frac{1}{x}\right)$ ...(*iii*)

Dividing (*ii*) by (*iii*), we get $\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{\frac{-y}{x^2}}{\frac{1}{x}}$

$\Rightarrow \quad \frac{p}{q} = -\frac{y}{x}$ or $px + qy = 0$

which is required partial differential equation (*iii*) We are given $z = f(x^2 - y^2)$

Differentiate partially w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x} = f'(x^2 - y^2)\frac{\partial}{\partial x}(x^2 - y^2) = 2xf'(x^2 - y^2)$ ...(

and $\frac{\partial z}{\partial y} = f'(x^2 - y^2)\frac{\partial}{\partial y}(x^2 - y^2) = -2yf'(x^2 - y^2)$ ...(

Dividing (*ii*) by (*iii*), we get

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{2xf'(x^2 - y^2)}{-2yf'(x^2 - y^2)} \Rightarrow y\frac{\partial z}{\partial x} = -x\frac{\partial z}{\partial y} \Rightarrow x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial x} = 0$$

which is required partial differential equation.

(*iv*) we are given $z = f(x^2 + 2y^2)$ ...

Differentiating partially w.r.t. x and w.r.t y, we get

$$\frac{\partial z}{\partial x} = f'(x^2 + 2y^2)\frac{\partial}{\partial x}(x^2 + 2y^2)$$

$$= 2xf'(x^2 + 2y^2)$$ ...(

and $\frac{\partial z}{\partial y} = f'(x^2 + 2y^2)\frac{\partial}{\partial y}(x^2 + 2y^2)$

$$= 4yf'(x^2 + y^2)$$ ...(i

On dividing (*ii*) by (*iii*), we get

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{2xf'(x^2 + y^2)}{4yf'(x^2 + y^2)} = \frac{x}{2y}$$

Cross-Multiply

$\Rightarrow$ $2y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$

which is required partial differential equation.

(*v*) We are given $z = x + y + f(xy)$ ...(i

Differentiating partially (*i*) w.r.t. x and w.r.t. y, we get

$$\frac{\partial z}{\partial x} = 1 + f'(xy)\frac{\partial}{\partial x}(xy)$$

$\Rightarrow$ $p = 1 + yf'(xy)$ ...(ii

and $\frac{\partial z}{\partial y} = 1 + f'(xy)\frac{\partial}{\partial y}(xy)$

$\Rightarrow \quad q = 1 + x f'(x y)$

From (*ii*) $p - 1 = y f'(x y)$ ...(*iii*)

and From (*iii*) $q - 1 = x f'(x y)$ ...(*iv*)

Divide (*iv*) by (*v*) we get $\dfrac{p-1}{q-1} = \dfrac{y}{x}$ cross multiply ...(*v*)

we get $x p - x = q y - y$

$\Rightarrow \quad p x - q y = x - y$ which is required partial differential equation.

(*vi*) We are given

$$z = x y + f(x^2 + y^2) \quad ...(i)$$

Differentiate partially w.r.t x and w.r.t y

we get $\dfrac{\partial z}{\partial x} = y + f'(x^2 + y^2)\dfrac{\partial}{\partial x}(x^2 + y^2)$

$\Rightarrow \quad p = y + 2 x f'(x^2 + y^2)$

$\Rightarrow \quad p - y = 2 x f'(x^2 + y^2)$ ...(*ii*)

and $\dfrac{\partial z}{\partial y} = x + f'(x^2 + y^2)\dfrac{\partial}{\partial y}(x^2 + y^2)$

$= x + f'(x^2 + y^2)(0 + 2 y)$

$\Rightarrow \quad q - x = f'(x^2 + y^2)(2 y)$ ...(*iii*)

On dividing (*ii*) by (*iii*), we get

$$\frac{p-y}{q-x} = \frac{2 x f'(x^2 + y^2)}{2 y f'(x^2 + y^2)} = \frac{x}{y}$$

Cross-multiply

$\Rightarrow \quad y(p - y) = x(q - x) \quad \Rightarrow \quad p y - q x = y^2 - x^2$

which is required differential equation.

(*vii*) We are given ...(*i*)

$$z = f(x) + e^y g(x)$$

Differentiate partially w.r.t. x and w.r.t. y ...(*ii*)

we get $\dfrac{\partial z}{\partial x} = f'(x) + e^y g'(x)$ ...(*iii*)

and $\dfrac{\partial z}{\partial y} = 0 + e^y g(x)$

Differentiate (*iii*) w.r.t. y ...(*iv*)

$$\frac{\partial^2 z}{\partial y^2} = e^y g(x)$$

From (*vi*) and (*vii*), we get

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=y\frac{\partial z}{\partial y}-x\frac{\partial z}{\partial x}$$

which is required partial differential equation

(*ix*) We are given $z=e^{ky}f(x-y)$ ...(*i*)

Differentiate (*i*) w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x}=e^{ky}\,f'(x-y)$ ...(*ii*)

and $\frac{\partial z}{\partial y}=k\,e^{ky}f(x-y)+e^{ky}\,f'(x-y)(-1)$ ...(*iii*)

Put values from (*i*) and (*ii*) in (*iii*)

We get $\frac{\partial z}{\partial y}=kz-\frac{\partial z}{\partial x}$

or $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=kz$ which is required partial differential equation.

(*x*) We are given

$z=f(x+ay)+g(x-ay)$ ...(*i*)

Differentiate (*i*) w.r.t. x and w.r.t. y

we get $\frac{\partial z}{\partial x}=f'(x+ay)(1+0)+g'(x-ay)(1-0)$

⇒ $\frac{\partial z}{\partial x}=f'(x+ay)+g'(x-ay)$ ...(*ii*)

and $\frac{\partial z}{\partial y}=f'(x+ay)(0+a)+g'(x-ay)(0-a)$

⇒ $\frac{\partial z}{\partial y}=a(f'(x+ay)-g'(x-ay))$ ...(*iii*)

Further Differentiate (*ii*) w.r.t x and (*iii*) w.r.t y

we get $\frac{\partial^2 z}{\partial x^2}=f''(x+ay)(1+0)+g''(x-ay)(1-0)$

$=f''(x+ay)+g''(x-ay)$ ...(*iv*)

and $\frac{\partial^2 z}{\partial y^2}=a(f''(x+ay)(0+a)-g''(x-ay)(0-a))$

$=a^2(f''(x+ay)+g''(x-ay))=a^2\frac{\partial^2 z}{\partial x^2}$ (using (*iv*))

⇒ $\frac{\partial^2 z}{\partial y^2}=a^2\frac{\partial^2 z}{\partial x^2}$

Which is required partial differential equation